Stability Analysis of Linear Time-Delay Systems with Two Incommensurate Delays

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Abstract The contribution focuses on the stability analysis of linear time-delay systems within the framework of the Lyapunov – Krasovskii functionals. The method used is based on the idea to check positive definiteness of the functionals exclusively on the specific Razumikhin-type set of functions. For the systems with incommensurate delays, it is proposed to use the modified functionals depend on the Lyapunov delay matrix related to a nominal system with commensurate delays. The method is applied for the estimation of the stability domains in the parameter space.

1 Introduction

The chapter is devoted to the application of the Lyapunov – Krasovskii functionals with the prescribed derivative, together with the Razumikhin approach, for stability analysis of time-delay systems. These functionals were first introduced in [9, 13], then further developed in [4, 5], and stated in the explicit form in [8]. The construction of the quadratic lower bounds for the functionals was one of the most important problems in their application: in [4], it was shown that the functional with a derivative prescribed as a negative definite quadratic form admits only the local cubic lower bound. The problem was solved in [8] by introducing the functionals of the complete type that admit the quadratic lower bounds, and for this reason can be effective in obtaining of the exponential estimates for the solutions [6] and the robustness bounds [7, 8]. The detailed account of the applications of the complete-type functionals is provided in book [6].

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The Lyapunov – Krasovskii functionals are determined by the so-called delay Lyapunov matrix that plays a key role in the application of the theory: To construct the functional, one needs to compute the Lyapunov matrix. Moreover, recently the approach that allows to analyze the stability directly through the Lyapunov matrix was appeared, see the work [2]. For the systems with commensurate delays, this matrix can be computed as the solution of the ordinary differential system with the special boundary conditions, see the semianalytic method in [3]. In contrast to that, when the delays are incommensurate, the only approximation schemes, like the piecewise linear one in [3], are available.

In papers [10, 11, 14], we apply the Lyapunov – Krasovskii functionals for stability analysis using a special Razumikhin-type condition. It turns out that the stability criterion can be expressed in terms of the fact that the functional admits the quadratic lower bound on the functions satisfying this condition. The main advantage of the approach is its constructiveness. Another advantage is that it allows to obtain both stability and instability necessary and sufficient conditions, thus arriving at the comprehensive stability picture.

The application of this stability approach for the systems with incommensurate delays faces the problem of computation of the Lyapunov matrices for such systems. To avoid this computational problem, in this contribution we propose the modification of the functional that consists in the replacement of the Lyapunov matrix in the functional with the one corresponding to the "close" system with commensurate delays. The "proximity" between the related system with commensurate delays and the original one is defined by the condition whether the time derivative of the modified functional is negative definite or not. Such modification allows to extend the results from [10, 11, 14] to the systems with incommensurate delays and to formulate the constructive procedure for the exponential stability and instability analysis of such systems which is the main contribution of the chapter.

The chapter is organized as follows. In the next section, the basic notations and the main concepts of the Lyapunov – Krasovskii approach are presented. In Section 3, the summary of works [10, 11, 14] concerning the systems with two arbitrary delays is given. It includes necessary and sufficient stability and instability conditions (Subsection 3.1), with the proofs in the Appendices, the description of the method for stability analysis based on the piecewise linear approximation scheme (Subsection 3.2), and the brief discussion of the convergence issue (Subsection 3.3).

Section 4 contains the main contribution of the chapter. It is dedicated to stability analysis of systems with two incommensurate delays, with the help of the approach presented in Section 3. In particular, Subsection 4.1 states the problem, Subsection 4.2 presents the modification of the functional that allows to deal with the systems with incommensurate delays. Subsection 4.3 is devoted to the main stability and instability results which are the modifications of the results of Subsection 3.1. The illustrative example provides the estimation of the stability domain in the parameter space for the scalar equation with two incommensurate delays, see Subsection 4.4. The conclusion ends the chapter.

2 Preliminaries

Here we present the basic concepts of the Lyapunov – Krasovskii functionals approach summarized from [6, 8].

In this chapter, we consider the time-delay system of the form

$$\dot{x}(t) = A_0x(t) + A_1x(t-h_1) + A_2x(t-h_2), \quad t \geqslant 0, \tag{1}$$

where $x \in \mathbb{R}^n$, A_j, $j = 0, 1, 2$, are the constant $n \times n$ matrices, and $0 < h_1 < h_2 = h$ are the delays, commensurate or not. Let $\varphi(\theta)$ be the piecewise continuous initial function, and $x(t, \varphi)$, or briefly $x(t)$, be the solution of system (1) such that $x(\theta, \varphi) = \varphi(\theta)$, $\theta \in [-h, 0]$. As usual, we denote by $x_t(\varphi)$, or briefly x_t, the segment of the trajectory: $\{x(t+\theta, \varphi) \mid \theta \in [-h, 0]\}$, and suppose that $\|\varphi\|_h = \sup_{\theta\in[-h,0]} \|\varphi(\theta)\|$.

We follow the classical stability concept given in [1, 6]: System (1) is said to be exponentially stable, if there exist $\sigma > 0$ and $\gamma \geqslant 1$ such that

$$\|x(t, \varphi)\| \leqslant \gamma e^{-\sigma t}\|\varphi\|_h, \quad t \geqslant 0.$$

The matrix $U(\tau)$ is called the Lyapunov matrix of system (1) associated with the symmetric matrix W [6] if it satisfies the equations

$$\begin{gathered} U'(\tau) = U(\tau)A_0 + U(\tau-h_1)A_1 + U(\tau-h_2)A_2, \ \tau \geqslant 0, \qquad (2) \\ U(-\tau) = U^T(\tau), \ \tau \geqslant 0, \\ U(0)A_0 + A_0^TU(0) + U(-h_1)A_1 + A_1^TU(h_1) + U(-h_2)A_2 + A_2^TU(h_2) = -W. \end{gathered}$$

According to [6], system (1) admits the unique Lyapunov matrix, if and only if it does not have the eigenvalues the sum of which is equal to zero. The latter condition is known as the Lyapunov condition.

The Lyapunov matrix determines the functional with the given derivative which is generally used for the exponential stability analysis of system (1). Let us first assign the quadratic form $w_0(x(t)) = x^T(t)Wx(t)$, then the quadratic functional, whose time derivative along the solutions of system (1) is equal to $-w_0(x(t))$, is of the form

$$\begin{aligned} v_0(x_t, U) = x^T(t)U(0)x(t) &+ 2x^T(t)\sum_{j=1}^{2}\int_{-h_j}^{0} U(-\theta-h_j)A_jx(t+\theta)d\theta \\ + \sum_{k=1}^{2}\sum_{j=1}^{2}\int_{-h_k}^{0} x^T(t+\theta_1)A_k^T&\left(\int_{-h_j}^{0} U(\theta_1+h_k-\theta_2-h_j)A_jx(t+\theta_2)d\theta_2\right)d\theta_1, \end{aligned} \tag{3}$$

here the Lyapunov matrix $U(\tau)$ is associated with W. If we now assign the functional

$$w(x_t) = x^T(t)W_0x(t) + \sum_{j=1}^{2} \int_{-h_j}^{0} x^T(t+\theta)W_jx(t+\theta)d\theta$$

with the symmetric matrices W_0, W_1, and W_2, then the functional

$$v(x_t,U) = v_0(x_t,U) + \sum_{j=1}^{2} \int_{-h_j}^{0} x^T(t+\theta)\big[(h_j+\theta)W_j\big]x(t+\theta)d\theta, \qquad (4)$$

where the Lyapunov matrix $U(\tau)$ is associated with $W = W_0 + h_1W_1 + h_2W_2$, is such that

$$\frac{d}{dt}v(x_t,U) = -w(x_t), \quad t \geqslant 0, \qquad (5)$$

along the solutions of system (1). The use of functional (4) is in line with the use of the complete-type functionals [6].

According to [6] (see p. 58, Example 2.1) and [4], functional (3) does not admit the quadratic lower bound. Nevertheless, in the next section we provide the stability results that enable to check the stability of system (1) with the help of this functional.

3 Synthesis of Razumikhin and Lyapunov – Krasovskii Approaches: Previous Results

In this section, we present the summary of contributions [10, 11, 14]. These papers introduce the new approach for stability analysis which is based on the idea to check positive definiteness of functional (3) or (4) on the special set of functions

$$S = \big\{\varphi : \|\varphi(\theta)\| \leqslant \|\varphi(0)\|,\ \theta \in [-h,0]\big\},$$

instead of the set of solutions as in [9]. This approach allows us to obtain necessary and sufficient conditions of exponential stability and instability and to analyze the stability of system (1) constructively.

3.1 Stability results

Our basic stability results [10, 14] are the following.

Theorem 1. *Given positive definite matrix W, system* (1) *is exponentially stable, if and only if there exists a functional $v_0(\varphi,U)$ such that*

1. $\frac{d}{dt}v_0(x_t,U) = -x^T(t)Wx(t)$ *along the solutions of system* (1)*;*

2. *there exists* $\mu > 0$ *such that* $v_0(\varphi, U) \geqslant \mu \|\varphi(0)\|^2$ *for every function* $\varphi \in S$.

Theorem 2. *Let system* (1) *satisfy the Lyapunov condition. Given positive definite matrix* W, *system* (1) *is unstable, if and only if there exists a functional* $v_0(\varphi, U)$ *such that*

1. $\dfrac{d}{dt} v_0(x_t, U) = -x^T(t) W x(t)$ *along the solutions of system* (1);
2. *there exists* $\mu > 0$ *and a function* $\varphi \in S$ *such that* $v_0(\varphi, U) \leqslant -\mu \|\varphi(0)\|^2$.

The proofs of Theorems 1 and 2 are provided in Appendices 1 and 2 respectively for the completeness.

Remark 1. Theorems 1 and 2 remain valid with functional (4) and statement (5), for some positive definite matrices W_0, W_1, and W_2, in their first conditions.

Remark 2. Theorems 1 and 2 remain valid with the set

$$S_k = \left\{ \varphi : \|\varphi^{(l)}(\theta)\| \leqslant \big(\|A_0\| + \|A_1\| + \|A_2\|\big)^l \|\varphi(0)\|, \ \theta \in [-h, 0], \ l = \overline{0, k+1} \right\}$$

instead of S, as follows from their proofs. Here $\varphi^{(l)}(\theta)$ stands for the l-th derivative of $\varphi(\theta)$, and k is any natural number. In Subsection 3.2, we use this fact with $k = 1$. In [10], the modification of the method with S_3 is considered.

3.2 Description of the Method for Stability Analysis

Here we present the constructive method for stability analysis based on Theorem 1. The point is to derive the lower bound for functional (3) or (4) using the estimations from the set S_1.

To this end, consider the partition of the intervals $[-h_2, -h_1]$ and $[-h_1, 0]$ into N_1 and N_2 equal parts respectively by the points

$$\theta_k^{(1)} = -k\Delta_1, \quad k = \overline{0, N_1}, \quad \Delta_1 = \frac{h_1}{N_1},$$
$$\theta_k^{(2)} = -h_1 - k\Delta_2, \quad k = \overline{0, N_2}, \quad \Delta_2 = \frac{h_2 - h_1}{N_2}.$$

Then, approximate an arbitrary vector function $\varphi \in S_1$ in each small interval by the linear function

$$l_k^{(j)}(\theta) = \varphi(\theta_k^{(j)}) \Big(1 + \frac{\theta}{\Delta_j}\Big) - \varphi(\theta_{k+1}^{(j)}) \frac{\theta}{\Delta_j}, \ \theta \in [-\Delta_j, 0],$$
$$k = \overline{0, N_j - 1}, \ j = 1, 2,$$

so that

$$\varphi(\theta + \theta_k^{(j)}) = l_k^{(j)}(\theta) + \eta_k^{(j)}(\theta), \quad \theta \in [-\Delta_j, 0], \tag{6}$$

where $\eta_k^{(j)}(\theta)$ is the error of such approximation on the interval $[\theta_{k+1}^{(j)}, \theta_k^{(j)}]$. Estimating the error, we obtain

$$\|\eta_k^{(j)}(\theta)\| \leqslant \frac{1}{2}\sqrt{n}\Big(\|A_0\| + \|A_1\| + \|A_2\|\Big)^2 \|\varphi(0)\|(\theta^2 - \theta\Delta_j), \qquad (7)$$
$$\theta \in [-\Delta_j, 0],\ k = \overline{0, N_j - 1},\ j = 1, 2.$$

To illustrate how to substitute approximation (6) into the functional, consider and transform one of its summands:

$$2\varphi^T(0)\int\limits_{-h_1}^{0} U(-\theta - h_1)A_1\varphi(\theta)d\theta$$
$$= 2\varphi^T(0)\sum_{k=1}^{N_1}\int\limits_{-\Delta_1}^{0} U\big(-\theta + \theta_{N_1-k+1}^{(1)}\big)A_1\varphi(\theta + \theta_{k-1}^{(1)})d\theta$$
$$= 2\varphi^T(0)\sum_{k=1}^{N_1}\Big[\int\limits_{-\Delta_1}^{0} U\big(-\theta + \theta_{N_1-k+1}^{(1)}\big)\Big(1 + \frac{\theta}{\Delta_1}\Big)d\theta A_1\varphi(\theta_{k-1}^{(1)})$$
$$-\int\limits_{-\Delta_1}^{0} U\big(-\theta+\theta_{N_1-k+1}^{(1)}\big)\frac{\theta}{\Delta_1}d\theta A_1\varphi(\theta_k^{(1)}) + \int\limits_{-\Delta_1}^{0} U\big(-\theta+\theta_{N_1-k+1}^{(1)}\big)A_1\eta_{k-1}^{(1)}d\theta\Big].$$

Dealing similarly with other summands and estimating the approximation error $\eta_k^{(j)}$ by (7), we obtain the following bound for the functional

$$v(\varphi, U) \geqslant \big(p^T\, \widehat{\varphi}^T\big)^T \mathbf{V}\begin{pmatrix} p \\ \widehat{\varphi}\end{pmatrix} = v_0(p, \widehat{\varphi}, \mathbf{h}, \mathbf{N}) - \delta(\mathbf{h}, \mathbf{N})\|p\|^2, \quad \varphi \in S_1. \quad (8)$$

Here $\mathbf{h} = (h_1, h_2)^T,\ \mathbf{N} = (N_1, N_2)^T,\ p = \varphi(0)$, and

$$\widehat{\varphi} = \Big(\varphi^T\big(\theta_1^{(1)}\big), \dots, \varphi^T\big(\theta_{N_1}^{(1)}\big), \varphi^T\big(\theta_1^{(2)}\big), \dots, \varphi^T\big(\theta_{N_2}^{(2)}\big)\Big)^T.$$

Bound (8) is the quadratic form of the vector $(p^T, \widehat{\varphi}^T)^T$, and the elements of the matrix $\mathbf{V}$ depend on the Lyapunov matrix. The expression for the bound naturally falls into two groups of summands: the first one, $v_0(p, \widehat{\varphi}, \mathbf{h}, \mathbf{N})$, presents functional (4) on the piecewise linear approximation, without taking the error into account, while the second one, $-\delta(\mathbf{h}, \mathbf{N})\|p\|^2$, includes all the summands with the approximation error for which estimation (7) is applied.

Let us rewrite the set S_1 in new notations

$$\widehat{S}_{\mathbf{N}} = \big\{\widehat{\varphi} : \|\varphi(\theta_k^{(j)})\| \leqslant \|p\|,\ k = \overline{1, N_j},\ j = 1, 2\big\},$$

and introduce the function

$$z(\mathbf{h},\mathbf{N}) = \min_{\substack{\widehat{\varphi}\in\widehat{S}_{\mathbf{N}} \\ \|p\|=1}} v_0(p,\widehat{\varphi},\mathbf{h},\mathbf{N}) - \delta(\mathbf{h},\mathbf{N}). \tag{9}$$

Since estimation (8) holds for every function $\varphi \in S_1$, the sufficiency of Theorem 1 can be rephrased in the following way.

Theorem 3. *If there exist* N_1 *and* N_2 *such that* $z(\mathbf{h},\mathbf{N}) > 0$, *then system* (1) *is exponentially stable.*

The result follows from Remark 2 and the fact that the sign of the minimum in (9) does not depend on $\|p\| \neq 0$. Theorem 3 shows the way of application of the method described.

Remark 3. The method can be applied for the stability analysis of system (1) with incommensurate delays h_1 and h_2, if the Lyapunov matrix is known. The only problem in this case is its computation. The modification of the method presented in the next section makes it possible to avoid computation of the Lyapunov matrix for the system with incommensurate delays.

Remark 4. On the basis of Theorem 2, by analogy with Theorem 3, we can formulate the following sufficient condition of instability: If there exist N_1 and N_2 such that

$$\tilde{z}(\mathbf{h},\mathbf{N}) = \min_{\substack{\widehat{\varphi}\in\widehat{S}_{\mathbf{N}} \\ \|p\|=1}} v_0(p,\widehat{\varphi},\mathbf{h},\mathbf{N}) + \delta(\mathbf{h},\mathbf{N}) < 0,$$

then system (1) is unstable.

3.3 Convergence Issue

In this section, we discuss an important property of the method we use, namely, the convergence. The convergence plays a key role in the application of the method: it guarantees that for every exponentially stable system of the form (1) there exists $\mathbf{N}$ such that Theorem 3 holds, i.e. the stability is ensured by our method.

Without loss of generality we can assume that $h_1 = \alpha_1\mathfrak{h}$, $h_2 = \alpha_2\mathfrak{h}$, where $\alpha_1, \alpha_2 > 0$; $\mathfrak{h} \geqslant 0$ is the basic delay. The convergence is based on the fact

$$\delta(\alpha_1\mathfrak{h}, \alpha_2\mathfrak{h}, N_1, N_2) \xrightarrow[N_1,N_2\to+\infty]{} 0 \tag{10}$$

that leads to the following statement.

Lemma 1. *Let system* (1) *be exponentially stable. Then there exist* $\widetilde{N}_1$ *and* $\widetilde{N}_2$ *such that*

$$z(\alpha_1\mathfrak{h}, \alpha_2\mathfrak{h}, N_1, N_2) > 0$$

for any $N_1 \geqslant \widetilde{N}_1$, $N_2 \geqslant \widetilde{N}_2$.

Suppose that system (1) is exponentially stable for $\mathfrak{h} \in (\bar{\mathfrak{h}}_1, \bar{\mathfrak{h}}_2)$, and it loses the property of exponential stability for the basic delays $\bar{\mathfrak{h}}_1$ and $\bar{\mathfrak{h}}_2$. Assume that $0 < \bar{\mathfrak{h}}_1 < \bar{\mathfrak{h}}_2 < +\infty$.

Let us fix $\tilde{\mathfrak{h}} \in (\bar{\mathfrak{h}}_1, \bar{\mathfrak{h}}_2)$, and find $\widetilde{N}_1$ and $\widetilde{N}_2$ for the system with the basic delay $\tilde{\mathfrak{h}}$ from Lemma 1. For $N_1 \geqslant \widetilde{N}_1$ and $N_2 \geqslant \widetilde{N}_2$ we define the sequences

$$\mathfrak{h}_{\mathbf{N}}^{(1)} = \sup_{\substack{\mathfrak{h} < \tilde{\mathfrak{h}} \\ z(\mathbf{h},\mathbf{N}) \leqslant 0}} \mathfrak{h}, \quad \mathfrak{h}_{\mathbf{N}}^{(2)} = \inf_{\substack{\mathfrak{h} > \tilde{\mathfrak{h}} \\ z(\mathbf{h},\mathbf{N}) \leqslant 0}} \mathfrak{h}. \tag{11}$$

It follows from (11) that $z(\mathbf{h}, \mathbf{N}) > 0$ for $\mathfrak{h} \in (\mathfrak{h}_{\mathbf{N}}^{(1)}, \mathfrak{h}_{\mathbf{N}}^{(2)})$, and thus,

$$(\mathfrak{h}_{\mathbf{N}}^{(1)}, \mathfrak{h}_{\mathbf{N}}^{(2)}) \subset (\bar{\mathfrak{h}}_1, \bar{\mathfrak{h}}_2),$$

due to continuity of function (9).

Theorem 4. *Sequences* (11) *converge, and*

$$\lim_{\substack{N_1 \to +\infty \\ N_2 \to +\infty}} \mathfrak{h}_{\mathbf{N}}^{(1)} = \bar{\mathfrak{h}}_1, \quad \lim_{\substack{N_1 \to +\infty \\ N_2 \to +\infty}} \mathfrak{h}_{\mathbf{N}}^{(2)} = \bar{\mathfrak{h}}_2.$$

The statement of Theorem 4 means that the stability interval of system (1), which is guaranteed by the method described in Section 3.2, tends to the exact one when $N_1 \to +\infty$ and $N_2 \to +\infty$.

Remark 5. The similar theorem, with the sequences

$$\tilde{\mathfrak{h}}_{\mathbf{N}}^{(1)} = \sup_{\substack{\mathfrak{h} < \tilde{\mathfrak{h}} \\ \tilde{z}(\mathbf{h},\mathbf{N}) \geqslant 0}} \mathfrak{h}, \quad \tilde{\mathfrak{h}}_{\mathbf{N}}^{(2)} = \inf_{\substack{\mathfrak{h} > \tilde{\mathfrak{h}} \\ \tilde{z}(\mathbf{h},\mathbf{N}) \geqslant 0}} \mathfrak{h},$$

where $\tilde{\mathfrak{h}}$ is a point from the instability interval of system (1), can be formulated for the instability case, as well.

4 Systems with Incommensurate Delays

In this section, we apply the results of the previous one for the stability analysis of the systems with incommensurate delays using the special modification of the functional.

4.1 Problem Formulation

Consider system (1) with $h_1 = 1$ and $h_2 = h$, where the delay $h > 1$ is an irrational number. Together with (1) introduce the following system

$$\dot{y}(t) = A_0 y(t) + A_1 y(t-1) + A_2 y(t-\hat{h}), \quad \hat{h} \in \mathbb{Q}. \tag{12}$$

Our aim is to find the conditions on the rational delay $\hat{h}$ such that the exponential stability of system (1) can be analyzed by the method described in Section 3.2, with the help of the modified functional which depends on the Lyapunov matrix of system (12).

4.2 Modified Functional

To analyze the stability of the system with incommensurate delays, we will use the functional $v(x_t, U_{\hat{h}})$, where $U_{\hat{h}}(\tau)$ is the Lyapunov matrix of system (12) associated with $W = W_0 + W_1 + hW_2$. The modified functional differs from functional (4) only by the Lyapunov matrix; it depends on the Lyapunov matrix $U_{\hat{h}}(\tau)$ for $\tau \in [-h, h]$. For the existence of the functional $v(x_t, U_{\hat{h}})$ we need the following assumption.

Assumption 1 *System* (12) *satisfies the Lyapunov condition.*

To apply the new functional for stability analysis of system (1), one needs to compute its time derivative along the solutions of this system. Introduce the matrix

$$\Delta U_{\hat{h}}(\tau) = U_{\hat{h}}(h-\tau) - U_{\hat{h}}(\hat{h}-\tau), \quad \tau \in [0, h],$$

and the functional

$$R(x_t, \Delta U_{\hat{h}}) = x^T(t) \left[A_2^T \Delta U_{\hat{h}}(0) + \left(\Delta U_{\hat{h}}(0)\right)^T A_2 \right] x(t) + 2x^T(t) A_2^T \times$$
$$\times \int_{-1}^{0} \Delta U_{\hat{h}}(\theta+1) A_1 x(t+\theta) d\theta + 2x^T(t) A_2^T \int_{-h}^{0} \Delta U_{\hat{h}}(\theta+h) A_2 x(t+\theta) d\theta.$$

Direct differentiation and use of properties (2) for the Lyapunov matrix $U_{\hat{h}}(\tau)$, in a similar way as in [6] (p. 40, the proof of Theorem 2.4), lead to the following lemma.

Lemma 2. *The time derivative of the functional* $v(x_t, U_{\hat{h}})$ *along the solutions of system* (1) *is of the form*

$$\frac{d}{dt} v(x_t, U_{\hat{h}}) = -w(x_t) + R(x_t, \Delta U_{\hat{h}}), \quad t \geqslant 0.$$

Our next purpose is to check when the obtained time derivative is negative definite. To do this, first let

$$M = \max_{\tau \in [0,h]} \|\Delta U_{\hat{h}}(\tau)\| < +\infty.$$

Direct estimation shows that the functional $R(x_t, \Delta U_{\hat{h}})$ admits the following upper bound

$$R(x_t, \Delta U_{\hat{h}}) \leqslant M\Big(\xi_0 \|x(t)\|^2 + \xi_1 \int\limits_{-1}^{0} \|x(t+\theta)\|^2 d\theta + \xi_2 \int\limits_{-h}^{0} \|x(t+\theta)\|^2 d\theta\Big),$$

where $\xi_0 = \|A_2\|\big(2 + \|A_1\| + h\|A_2\|\big)$, $\xi_1 = \|A_1\|\|A_2\|$, $\xi_2 = \|A_2\|^2$, and we arrive at the following lemma.

Lemma 3. *If the following inequalities hold*

$$\xi_0 M < \lambda_{\min}(W_0), \quad \xi_1 M \leqslant \lambda_{\min}(W_1), \quad \xi_2 M \leqslant \lambda_{\min}(W_2), \tag{13}$$

then the time derivative of the functional $v(x_t, U_{\hat{h}})$ along the solutions of system (1) *is negative definite.*

In the rest of the chapter we suppose that the condition of Lemma 3 is satisfied:

Assumption 2 *Inequalities* (13) *hold.*

Remark 6. Lemma 3 shows why the modification of functional (3) can not be used for stability analysis of the system with incommensurate delays: we can not guarantee the negative definiteness of its time derivative.

4.3 Stability Results

Here we present the results that allow to analyze the exponential stability and instability of system (1) with the delays $h_1 = 1$ and $h_2 = h \in \mathbb{R} \setminus \mathbb{Q}$. These results are the modifications of Theorems 1 and 2 with the functional presented in the previous section.

Theorem 5. *Let Assumptions 1 and 2 hold. Given positive definite matrices W_0, W_1 and W_2, system* (1) *is exponentially stable, if and only if there exists $\mu > 0$ such that*

$$v(\varphi, U_{\hat{h}}) \geqslant \mu \|\varphi(0)\|^2 \quad \text{for every} \quad \varphi \in S.$$

Proof. The proof is the modification of the proof of Theorem 1, see Appendix 1.
Necessity. As in the necessity part of Theorem 1, we first take a function $\varphi \in S$ and obtain that there exists $\delta > 0$ such that for the solution of system (1) the following estimation holds

$$\|x(t, \varphi)\| \geqslant \frac{\|\varphi(0)\|}{2}, \quad t \leqslant \delta,$$

the value of δ depends only on system (1) and does not depend on $\hat{h}$.

Then, since system (1) is exponentially stable and Assumption 2 holds, we have

$$v(\varphi, U_{\hat{h}}) = \int_0^{+\infty} \big(w(x_t) - R(x_t, \Delta U_{\hat{h}})\big)dt \geqslant \int_0^{+\infty} \Big[\big(\lambda_{\min}(W_0) - \xi_0 M\big)\|x(t)\|^2$$

$$+\big(\lambda_{\min}(W_1) - \xi_1 M\big)\int_{-1}^{0} \|x(t+\theta)\|^2 d\theta + \big(\lambda_{\min}(W_2) - \xi_2 M\big)\int_{-h}^{0} \|x(t+\theta)\|^2 d\theta\Big] dt$$

$$\geqslant \big(\lambda_{\min}(W_0) - \xi_0 M\big)\int_0^{\delta} \|x(t)\|^2 dt \geqslant \mu\|\varphi(0)\|^2,$$

where $\mu = \big(\lambda_{\min}(W_0) - \xi_0 M\big)\dfrac{\delta}{4} > 0$, $x(t)$ denotes the solution $x(t,\varphi)$ for simplicity. The necessity is proved.
Sufficiency. As in the sufficiency of Theorem 1, suppose, by contradiction, that system (1) is not exponentially stable. It means that there exists the sequence $\{t_k\}_{k=1}^{\infty}$, such that $t_k - t_{k-1} \geqslant h$, $t_k \xrightarrow[k\to+\infty]{} +\infty$, and $\|x(t_k)\| \geqslant \beta > 0$, where $x(t)$ is the solution of system (1).

The first case we consider is that there exists $G > 0$ such that $\|x(t)\| \leqslant G$, $t \geqslant -h$. In this case, as it was obtained in the proof of Theorem 1, for the solution $x(t)$ we can write

$$\|x(t)\| \geqslant \frac{\beta}{2}, \quad t \in [t_k, t_k + \tau], \quad \tau = \min\Big\{\frac{\beta}{2KG}; h\Big\},$$

for every k. Then, the estimation $v(\varphi, U_{\hat{h}}) \leqslant \eta G^2$, where $\eta = \text{const} > 0$, holds, and by analogy with Theorem 1, we have

$$v(\varphi, U_{\hat{h}}) = v(x_t, U_{\hat{h}}) + \int_0^{t} \big(w(x_s) - R(x_s, \Delta U_{\hat{h}})\big)ds \geqslant -\eta G^2 \tag{14}$$

$$+\sum_{k=1}^{N(t)} \int_{t_k}^{t_k+\tau} \Big[\big(\lambda_{\min}(W_0) - \xi_0 M\big)\|x(s)\|^2 + \big(\lambda_{\min}(W_1) - \xi_1 M\big)\int_{-1}^{0} \|x(s+\theta)\|^2 d\theta$$

$$+\big(\lambda_{\min}(W_2) - \xi_2 M\big)\int_{-h}^{0} \|x(s+\theta)\|^2 d\theta\Big] ds$$

$$\geqslant -\eta G^2 + \big(\lambda_{\min}(W_0) - \xi_0 M\big)\frac{\beta^2\tau}{4} N(t) \xrightarrow[t\to+\infty]{} +\infty,$$

where $N(t) \xrightarrow[t\to+\infty]{} +\infty$ is the number of intervals $[t_k, t_k + \tau]$ contained in $[0, t]$. The intervals $[t_k, t_k + \tau]$ do not intersect with each other for different k. A contradiction.

At the second case, when the solution $x(t)$ is not uniformly bounded, the proof is similar to that of Theorem 1. □

Theorem 6. *Let Assumptions 1 and 2 hold, and let system* (1) *satisfy the Lyapunov condition. Given positive definite matrices* W_0, W_1 *and* W_2, *system* (1) *is unstable, if and only if there exists* $\mu > 0$ *and a function* $\varphi \in S$ *such that*

$$v(\varphi, U_{\hat{h}}) \leqslant -\mu \|\varphi(0)\|^2.$$

Proof. The proof differs from that of Theorem 2 only by formula (16); throughout the proof, it should be replaced with

$$v(\tilde{x}_T, U_{\hat{h}}) - v(\tilde{x}_0, U_{\hat{h}}) = -\int\limits_0^T \big(w(\tilde{x}_t) - R(\tilde{x}_t, \Delta U_{\hat{h}})\big)dt,$$

that, by analogy with (14), leads to

$$v(\tilde{x}_T, U_{\hat{h}}) - v(\tilde{x}_0, U_{\hat{h}}) \leqslant -\big(\lambda_{\min}(W_0) - \xi_0 M\big) \int\limits_0^T \|\tilde{x}(t)\|^2 dt.$$

The rest of the proof repeats the proof of Theorem 2. □

Theorems 5 and 6 give the constructive way of stability and instability analysis of system (1) with incommensurate delays. Since the only difference between functionals $v(x_t, U)$ and $v(x_t, U_{\hat{h}})$ is the Lyapunov matrix, the only change we should do to apply the method described in Section 3.2 here is to put the corresponding Lyapunov matrix into the coefficients of the quadratic form in (8); the same can be said for the instability case. As $U_{\hat{h}}(\tau)$ is the Lyapunov matrix of system (12) with commensurate delays, the exact method for its construction is known (see [3, 6]). Therefore, to analyze the stability of system (1) with incommensurate delays we do not need to compute the Lyapunov matrix of this system.

4.4 Example

Here we provide the example that illustrates the application of the presented theory for the estimation of the stability domain in the parameter space, for the scalar equation with two incommensurate delays.

Consider the equation

$$\dot{x}(t) = -x(t) + bx(t-1) + cx(t-h), \tag{15}$$

where $b, c \in \mathbb{R}$, $h = \sqrt{5}/2$. Our aim is to analyze the exponential stability of equation (15).

First, let $b = 1$, $c = -1$. Choose $\hat{h} = 23/20 = 1.15$, and fix $N_1 = 10$, $N_2 = 2$. With these parameters, the method ensures the exponential stability of the equation.

Further, let us find the stability domain of the equation in the space of parameters b and c. To this end, we first apply the D-subdivision technique. Substituting $s = j\omega$, where $\omega \geqslant 0$ and j is the imaginary unit, $j^2 = -1$, into the characteristic equation of (15)

$$s + 1 - be^{-s} - ce^{-sh} = 0,$$

we obtain the D-curves as $b+c = 1$, and as the parametric function of ω in the form

$$b = \frac{1}{\cos\omega} + \frac{(\sin\omega + \omega\cos\omega)\cos\omega h}{\sin(\omega(h-1))\cos\omega},$$
$$c = -\frac{\sin\omega + \omega\cos\omega}{\sin(\omega(h-1))}.$$

On Figure 1, the boundaries of the partition are depicted by the curves. It is clear that the region containing the zero point is the stability region, and all other regions containing the points with $b = 0$ or $c = 0$ are the instability ones. We have no information about the stability of the equation in the remaining domains.

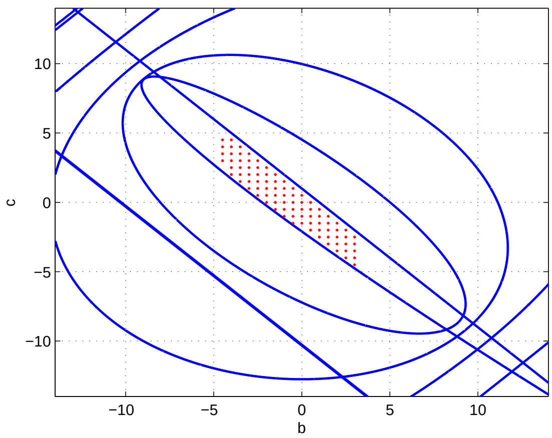

Fig. 1 Exponential stability domain of equation (15) in the space of parameters b and c

Let us now apply the results presented in the previous subsection. Fix $\hat{h} = 28/25 \approx 1.12$. The points on Figure 1 depict the stability points obtained by our method with the different values N_1 and N_2 smaller than or equal to $N_1 = 150$ and $N_2 = 70$.

These results conform to that obtained by the D-subdivision method. However, they show that even reasonably large values N_1 and N_2 are not enough to obtain all the stability region by our method. The second problem is that Assumption 2 does not hold throughout the stability region for chosen $\hat{h}$. The difficulty lies in the fact that the quantity of delays in the auxiliary system constructed for the computation of the Lyapunov matrix increases, in general, when $\hat{h}$ tends to h.

Nevertheless, asymptotic relation (10) remains valid in the modified case, with the Lyapunov matrix of system (12), for every fixed h and $\hat{h}$ satisfying Assumptions 1 and 2, and for this reason the stability domain obtained by our method should converge to the exact one as before, when $N_1, N_2 \to +\infty$, $\hat{h} \to h$.

5 Conclusion

In the chapter, theory of the Lyapunov – Krasovskii functionals, along with the Razumikhin approach, is applied for the stability analysis of linear system with two incommensurate delays. Necessary and sufficient stability and instability conditions are obtained with the help of the modified functional, that differs from the one that is usually used by the Lyapunov matrix: this matrix corresponds now to a system with commensurate delays. The modified method allows to analyze the stability constructively and remains convergent.

Appendix 1. Proof of Theorem 1

Necessity. This part of the proof is based on the proof of the main result in [4]. The first statement of the theorem holds for the functional $v_0(x_t, U)$ of the form (3). To prove the second one, take an arbitrary function $\varphi \in S$, and denote $\alpha = \|\varphi\|_h = \|\varphi(0)\|$.

For the solution of system (1), by Gronwall's lemma, we obtain

$$\|x(t,\varphi)\| \leqslant N(t) = \alpha K_1 e^{Kt},$$

where $K = \|A_0\| + \|A_1\| + \|A_2\|$, $K_1 = 1 + \|A_1\|h_1 + \|A_2\|h_2$. Hence,

$$\|\dot{x}(t,\varphi)\| \leqslant KN(t) \leqslant KN(\delta), \quad t \leqslant \delta,$$

for any $\delta > 0$, and $\|x(t,\varphi) - x(0,\varphi)\| \leqslant KN(\delta)t, \;\; t \leqslant \delta$. Choose δ so that $KN(\delta) = \alpha/(2\delta)$. Then, $\|x(t,\varphi)\| \geqslant \|\varphi(0)\| - \delta KN(\delta) = \|\varphi(0)\|/2, \; t \leqslant \delta$.

System (1) is exponentially stable, therefore,

$$v_0(\varphi, U) = \int\limits_0^{+\infty} x^T(t,\varphi) W x(t,\varphi) dt \geqslant \lambda_{\min}(W) \int\limits_0^{\delta} \|x(t,\varphi)\|^2 dt \geqslant \mu \|\varphi(0)\|^2,$$

where $\mu = \dfrac{\lambda_{\min}(W)\delta}{4} > 0$, and the proof of the necessity part is complete. Let us note that, in contrast to [4], the constant δ here does not depend on the initial function φ.
Sufficiency. Let system (1) be not exponentially stable. Then, there exists a sequence $\{t_k\}_{k=1}^{\infty}$, such that $t_k - t_{k-1} \geqslant h$, $t_k \xrightarrow[k\to+\infty]{} +\infty$, $\|x(t_k)\| \geqslant \beta > 0$.

At first suppose the solution $x(t)$ to be uniformly bounded: let there exists $G > 0$ such that $\|x(t)\| \leqslant G, \quad t \geqslant -h$. Hence, $\|\dot{x}(t)\| \leqslant KG, \quad t \geqslant 0$, where $K = \|A_0\| + \|A_1\| + \|A_2\|$, and

$$\|x(t) - x(t_k)\| \leqslant KG(t - t_k) \leqslant KG\tau, \quad t \in [t_k, t_k + \tau], \quad \tau > 0.$$

Choose $\tau = \min\Big\{\dfrac{\beta}{2KG}; h\Big\}$, then

$$\|x(t)\| \geqslant \|x(t_k)\| - KG\tau \geqslant \frac{\beta}{2}, \;\; t \in [t_k, t_k + \tau],$$

for every k. Let $N(t)$ be the number of intervals $[t_k, t_k + \tau] \subset [0, t]$; these intervals do not intersect with each other by definition of τ, and $N(t) \xrightarrow[t\to+\infty]{} +\infty$. Therefore,

$$\int_0^t x^T(s)Wx(s)ds \geqslant \sum_{k=1}^{N(t)} \int_{t_k}^{t_k+\tau} x^T(s)Wx(s)ds$$
$$\geqslant \lambda_{\min}(W)\frac{\beta^2\tau}{4}N(t) \xrightarrow[t\to+\infty]{} +\infty.$$

Since the functional $v_0(x_t, U)$ is bounded when the solution is bounded, we obtain the contradiction:

$$v_0(\varphi, U) = v_0(x_t, U) + \int_0^t x^T(s)Wx(s)ds \xrightarrow[t\to+\infty]{} +\infty.$$

Let us now assume that the solution $x(t)$ is not uniformly bounded. It means that the sequence $\{t_k\}_{k=1}^{\infty}$ can be chosen so that

$$\|x(t_k)\| = \max_{t \leqslant t_k} \|x(t)\| \xrightarrow[k\to+\infty]{} +\infty.$$

Such a choice results in $x_{t_k} \in S$ for every k, and

$$v_0(\varphi, U) = v_0(x_{t_k}, U) + \int_0^{t_k} x^T(s)Wx(s)ds \geqslant \mu\|x(t_k)\|^2 \xrightarrow[k\to+\infty]{} +\infty.$$

We obtain the contradiction that finishes the proof. $\square$

Appendix 2. Proof of Theorem 2

Necessity. Since system (1) satisfies the Lyapunov condition, there exists functional $v_0(x_t, U)$ of the form (3) (see [4]), for which the first statement of the theorem is true. Let us prove the second statement.

We first suppose that $\bar{\lambda} = \alpha > 0$ is the real eigenvalue of system (1). Then, the system has the solution $\tilde{x}(t) = e^{\alpha t}c$, where $c \in \mathbb{R}^n$, $c \neq \mathbf{0}$. Since $\tilde{x}(t)$ is the increasing function, $\tilde{x}_0 \in S$. On the other hand, we have

$$v_0(\tilde{x}_T, U) - v_0(\tilde{x}_0, U) = -\int\limits_0^T \tilde{x}^T(t)W\tilde{x}(t)dt = -\frac{1}{2\alpha}(e^{2\alpha T} - 1)\, c^T W c, \quad (16)$$

where $T = \text{const} > 0$. Since $\tilde{x}(T + \theta) = e^{\alpha T}\tilde{x}(\theta)$, $\theta \in [-h, 0]$, it follows that $v_0(\tilde{x}_T, U) = e^{2\alpha T} v_0(\tilde{x}_0, U)$, so (16) results in

$$v_0(\tilde{x}_0, U) = -\frac{1}{2\alpha}c^T W c \leqslant -\frac{\lambda_{\min}(W)}{2\alpha}\|c\|^2 = -\mu\|\tilde{x}(0)\|^2,$$

where $\mu = \dfrac{\lambda_{\min}(W)}{2\alpha} > 0$. The necessity is proved for $\bar{\lambda} \in R$.

We now turn to the case $\bar{\lambda} = \alpha + i\beta$, where $\alpha > 0$, $\beta \neq 0$. Let $c = c_1 + ic_2$ be the eigenvector corresponding to $\bar{\lambda}$, here $c_1, c_2 \in \mathbb{R}^n$. Choose $T = 2\pi/|\beta|$ and consider the T-periodic vector function $\psi(t) = \cos\beta t\, c_1 - \sin\beta t\, c_2$. Then, $e^{\alpha t}\psi(t)$ is the real part of $e^{\bar{\lambda} t}c$, and, therefore, is the solution of system (1). Since the system is time-invariant, function $\tilde{x}(t) = e^{\alpha(t+\bar{t})}\psi(t + \bar{t})$ is also the solution for every $\bar{t}$. Choose $\bar{t} \in [h, h + T]$ from the condition

$$\|\psi(\bar{t})\| = \max_{t\in[0,h+T]} \|\psi(t)\|,$$

such value of $\bar{t}$ exists due to continuity and periodicity of $\psi(t)$. Hence, $\|\tilde{x}(\theta)\| \leqslant \|\tilde{x}(0)\|$, $\theta \in [-h, 0]$, and therefore, $\tilde{x}_0 \in S$. Additionally, as in the first case, $v_0(\tilde{x}_T, U) = e^{2\alpha T} v_0(\tilde{x}_0, U)$.

Again consider the first equality in (16) and estimate its right-hand side. To this end, first note that $\tilde{x}(t) = e^{\alpha(t+\bar{t})}\big(\cos(\beta t)\xi_1 - \sin(\beta t)\xi_2\big)$, where $\xi_1 = \cos(\beta\bar{t})c_1 - \sin(\beta\bar{t})c_2 = \psi(\bar{t})$, $\xi_2 = \sin(\beta\bar{t})c_1 + \cos(\beta\bar{t})c_2$. Then,

$$\int\limits_0^T \tilde{x}^T(t)W\tilde{x}(t)dt \geqslant \lambda_{\min}(W)e^{2\alpha\bar{t}}\Bigg[\int\limits_0^T e^{2\alpha t}\cos^2(\beta t)dt\,\|\xi_1\|^2 + \int\limits_0^T e^{2\alpha t}\sin^2(\beta t)dt\,\|\xi_2\|^2 - \int\limits_0^T e^{2\alpha t}\sin(2\beta t)dt\,\xi_1^T\xi_2\Bigg].$$

Calculating directly all the integrals, using Cauchy – Bunyakovsky inequality for the term $\xi_1^T\xi_2$ and taking into account the fact that $\|\tilde{x}(0)\| = e^{\alpha\bar{t}}\|\xi_1\|$, we obtain

$$\int\limits_0^T \tilde{x}^T(t)W\tilde{x}(t)dt \geqslant \lambda_{\min}(W)e^{2\alpha\bar{t}}\frac{(e^{2\alpha T}-1)}{4\alpha(\alpha^2+\beta^2)}\Big[(\alpha^2+\beta^2)\|\xi_1\|^2 + (\alpha\|\xi_1\| - |\beta|\|\xi_2\|)^2\Big] \geqslant \mu(e^{2\alpha T}-1)\|\tilde{x}(0)\|^2,$$

where $\mu = \lambda_{\min}(W)/4\alpha > 0$. Combining the latter estimate with (16), we have $v_0(\tilde{x}_0, U) \leqslant -\mu\|\tilde{x}(0)\|^2$ for $\tilde{x}_0 \in S$, as required.

Sufficiency. Let us take the nontrivial initial function $\varphi \in S$ such that $v_0(\varphi, U) \leqslant -\mu\|\varphi(0)\|^2$. Condition $\varphi \in S$ implies $\|\varphi\|_h = \|\varphi(0)\|$, so $v_0(\varphi, U) \leqslant -\mu\|\varphi\|_h^2$.

Substituting the solution of system (1) corresponding to the function φ into functional $v_0(x_t, U)$ we obtain

$$v_0(x_t(\varphi), U) = v_0(\varphi, U) - \int\limits_0^t x^T(s,\varphi)Wx(s,\varphi)ds \leqslant -\mu\|\varphi\|_h^2. \tag{17}$$

Hence, $\mu\|\varphi\|_h^2 \leqslant |v_0(x_t, U)| \leqslant \eta\|x_t(\varphi)\|_h^2$, where $\eta = \text{const} > 0$, and finally,

$$\|x_t(\varphi)\|_h \geqslant \sqrt{\frac{\mu}{\eta}}\|\varphi\|_h > 0, \tag{18}$$

where the last expression we denote by β.

Let us prove that the solution $x(t, \varphi)$ is unstable. Conversely, suppose that there exists $G > 0$ such that $\|x(t,\varphi)\| \leqslant G$, $t \geqslant 0$. Then, $\|\dot{x}(t,\varphi)\| \leqslant KG$, where $K = \|A_0\| + \|A_1\| + \|A_2\|$. From (18) we have that there exists the sequence $\{t_k\}_{k=1}^{\infty}$, such that $t_k - t_{k-1} \geqslant h$, $t_k \xrightarrow[k\to+\infty]{} +\infty$, and $\|x(t_k,\varphi)\| \geqslant \beta > 0$. As in the proof of the sufficiency of Theorem 1, we can show that

$$\int\limits_0^t x^T(s,\varphi)Wx(s,\varphi)ds \xrightarrow[t\to+\infty]{} +\infty,$$

so, according to (17), $v_0(x_t, U) \xrightarrow[t\to+\infty]{} -\infty$, which contradicts the assumption that the solution $x(t,\varphi)$ is uniformly bounded. The theorem is proved. $\square$

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